Integral Means Inequalities for Fractional Deriatives Some General Subclasses of Analytic Functions (Inequalities in Univalent Function Theory and Its Applications)

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Integral Means Inequalities for Fractional Derivatives of Some General Subclasses of Analytic Functions

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Abstract

Integral means inequalities are obtained for the fractional derivatives of order of $p+\lambda$ $(0 \leq p \leq n;\ 0 \leq \lambda < 1)$ of functions belonging to certain general subclasses of analytic functions. Relevant connections with various known integral means inequalities are also pointed out.

Key words and phrases. Integral means inequalities, fractional derivatives, analytic functions, univalent functions, extreme points, subordination.
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1. Introduction, Definitions, and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are *analytic* in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{A}(n)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ of functions which are *univalent* in $\mathcal{U}$, and by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the subclasses of $\mathcal{T}(n)$ consisting of functions which are, respectively, *starlike of order* α $(0 \leqq \alpha < 1)$ and *convex of order* α $(0 \leqq \alpha < 1)$ in $\mathcal{U}$. The classes $\mathcal{A}(n)$, $\mathcal{T}(n)$, $\mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$ were investigated by Chatterjea [1](and Srivastava *et al.* [9]). In particular, the subclasses:

$$\mathcal{T} := \mathcal{T}(1), \quad \mathcal{T}^*(\alpha) := \mathcal{T}_\alpha(1), \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{C}_\alpha(1)$$

were considered earlier by Silverman [7].

Next, following the work of Sekine and Owa [4], we denote by $\mathcal{A}(n,\vartheta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\, a_k z^k \quad (\vartheta \in \mathbb{R};\ a_k \geqq 0;\ n \in \mathbb{N}). \tag{1.1}$$

Finally, the subclasses $\mathcal{T}(n,\vartheta)$, $\mathcal{T}^*_\alpha(n,\vartheta)$, and $\mathcal{C}_\alpha(n,\vartheta)$ of the class $\mathcal{A}(n,\vartheta)$ are defined in the same way as the subclasses $\mathcal{T}(n)$, $\mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$ of the class $\mathcal{A}(n)$.

We begin by recalling the following useful characterizations of the function classes $\mathcal{T}^*_\alpha(n,\vartheta)$ and $\mathcal{C}_\alpha(n,\vartheta)$(see Sekine and Owa [4]).

Lemma 1. *A function $f(z) \in \mathcal{A}(n,\vartheta)$ of the form* (1.1) *is in the class $\mathcal{T}^*_\alpha(n,\vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.2}$$

Lemma 2. *A function $f(z) \in \mathcal{A}(n,\vartheta)$ of the form* (1.1) *is in the class $\mathcal{C}_\alpha(n,\vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k\,(k-\alpha)\, a_k \leqq 1-\alpha \qquad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.3}$$

Motivated by the equalities in (1.2) and (1.3) above, Sekine *et al.* [6] defined a general subclass $\mathcal{A}(n; B_k, \vartheta)$ of the class $\mathcal{A}(n,\vartheta)$ consisting of functions $f(z)$ of the form (1.1), which satisfy the following inequality:

$$\sum_{k=n+1}^{\infty} B_k a_k \leqq 1 \quad (B_k > 0;\ n \in \mathbb{N}).$$

It is easy to verify each of the following classifications:

$$\mathcal{A}(n;k,\vartheta) = \mathcal{T}_0^*(n,\vartheta) =: \mathcal{T}^*(n,\vartheta) = \mathcal{T}(n,\vartheta),$$

$$\mathcal{A}\left(n;\frac{k-\alpha}{1-\alpha},\vartheta\right) = \mathcal{T}_\alpha^*(n,\vartheta) \quad (0 \leqq \alpha < 1),$$

and

$$\mathcal{A}\left(n;\frac{k(k-\alpha)}{1-\alpha},\vartheta\right) = \mathcal{C}_\alpha(n,\vartheta) \quad (0 \leqq \alpha < 1).$$

As a matter of fact, Sekine *et al.* [6] also obtained each of the following basic properties for the general classes $\mathcal{A}(n;B_k,\vartheta)$.

Theorem 1. *$\mathcal{A}(n;B_k,\vartheta)$ is the convex subfamily of the class $\mathcal{A}(n,\vartheta)$.*

Theorem 2. *Let*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{e^{i(k-1)\vartheta}}{B_k}z^k \tag{1.4}$$
$$(k = n+1, n+2, n+3, \dots;\ n \in \mathbb{N}).$$

Then, $f \in \mathcal{A}(n;B_k,\vartheta)$ if and only if $f(z)$ can be expressed as

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z),$$

where

$$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1 \quad (\lambda_1 \geqq 0;\ \lambda_k \geqq 0;\ n \in \mathbb{N}).$$

Corollary 1. *The extreme points of the class $\mathcal{A}(n;B_k,\vartheta)$ are the functions $f_1(z)$ and $f_k(z)$ $(k \geqq n+1;\ n \in \mathbb{N})$ given by* (1.4).

Applying the concepts of extreme points, fractional calculus, and subordination, Sekine *et al.* [6] obtained several integral means inequalities for higher-order fractional derivatives and fractional integral of functions belonging to the general classes $\mathcal{A}(n;B_k,\vartheta)$. Subsequently, Sekine and Owa [5] discussed the weakening of the hypotheses for B_k in those results by Sekine *et al.* [6]. In this paper, we investigate the integral means inequalities for the fractional derivatives of $f(z)$ of a general order $p+\lambda$ $(0 \leqq p \leqq n;\ 0 \leqq \lambda < 1)$ of functions $f(z)$ belonging to the general classes $\mathcal{A}(n; B_k, \vartheta)$.

We shall make use of the following definitions of fractional derivatives (*cf.* Owa [3]; see also Srivastava and Owa [8]).

Definition 1. The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta \quad (0 \leqq \lambda < 1), \tag{1.5}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the *fractional derivative of order* $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

It readily follows from (1.5) in Definition 1 that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leqq \lambda < 1). \tag{1.6}$$

We shall also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathcal{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathcal{U}$ if there exists a function $w(z)$, analytic in $\mathcal{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathcal{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Theorem 3 (Littlewood [2]). *If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with*

$$g(z) \prec f(z),$$

then

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \qquad (\mu > 0;\ 0 < r < 1).$$

2. The Main Integral Means Inequalities

Theorem 4. *Suppose that $f(z) \in \mathcal{A}(n; k^{p+1}B_k, \vartheta)$ and that*

$$\frac{(h+1)^q B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \leqq B_k \quad (k \geqq n+1)$$

for some $h \geqq n$, $0 \leqq \lambda < 1$, and $0 \leqq q \leqq p \leqq n$. Also let the function $f_{h+1}(z)$ be defined by

$$f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} z^{h+1} \quad \left(f_{h+1} \in A\left(n; k^{q+1}B_k, \vartheta\right)\right). \tag{2.1}$$

Then, for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^{p+\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{p+\lambda} f_{h+1}(z)\right|^{\mu} d\theta \qquad (0 \leqq \lambda < 1;\ \mu > 0).$$

Proof. By virtue of the fractional derivative formula (1.6) and Definition 2, we find from (1.1) that

$$\begin{aligned} &D_z^{p+\lambda} f(z) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} a_k\, z^{k-1}\right) \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1}\right) \end{aligned}$$

where

$$\Phi(k) := \frac{\Gamma(k-p)}{\Gamma(k+1-\lambda-p)} \qquad (0 \leqq \lambda < 1;\ k \geqq n+1;\ n \in \mathbb{N}).$$

Since $\Phi(k)$ is a *decreasing* function of k, we have

$$0 < \Phi(k) \leqq \Phi(n+1) = \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}$$
$$(0 \leqq \lambda < 1;\ k \geqq n+1;\ n \in \mathbb{N}).$$

Similarly, from (2.1), (1.6) and Definition 2, we obtain, for $0 \leqq \lambda < 1$,

$$D_z^{p+\lambda} f_{h+1}(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h\right).$$

For $z = re^{i\vartheta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1}\right|^{\mu} d\theta$$
$$\leqq \int_0^{2\pi} \left| 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h\right|^{\mu} d\theta$$
$$(0 \leqq \lambda < 1;\ \mu > 0).$$

Thus, by applying Theorem 3, it would suffice to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1}$$
$$\prec 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h. \quad (2.2)$$

Indeed, by setting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\Phi(k)a_k z^{k-1}$$
$$= 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} \{w(z)\}^h,$$

we find that

$$\{w(z)\}^h = \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{e^{ih\vartheta}\Gamma(h+2)} \cdot \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta}\frac{k!}{(k-p-1)!}\Phi(k)a_k z^k$$

which readily yields $w(0) = 0$.

Therefore, we have

$$
\begin{aligned}
&|w(z)|^h \\
&\leqq \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}\Phi(k)a_k|z|^{k-1} \\
&\leqq |z|^n \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \Phi(n+1) \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k \\
&= |z|^n \frac{(h+1)^{q+1}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k \\
&= |z|^n \frac{(h+1)^{q}B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}a_k \\
&\leqq |z|^n \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!}B_k a_k \\
&\leqq |z|^n \sum_{k=n+1}^{\infty} k^{p+1}B_k a_k \leqq |z|^n < 1 \quad (n \in \mathbb{N}),
\end{aligned}
\tag{2.3}
$$

by means of the hypothesis of Theorem 4.

In light of the last inequality in (2.3) above, we have the subordination (2.2), which evidently proves Theorem 4.

3. Remarks and Observations

First of all, in its special case when $p = q = 0$, Theorem 4 readily yields

Corollary 2(*cf.* Sekine and Owa [5, Theorem 6]). *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$
\frac{B_{h+1}\Gamma(h+2-\lambda)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+2-\lambda)} \leqq B_k \quad (k \geqq n+1)
$$

for some $h \geqq n$ *and* $0 \leqq \lambda < 1$. *Also let the function* $f_{h+1}(z)$ *be defined by*

$$
f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)B_{h+1}} z^{h+1} \quad (f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)). \tag{3.1}
$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^\lambda f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^\lambda f_{h+1}(z)\right|^\mu d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0). \tag{3.2}$$

A *further* consequence of Corollary 2 when $h = n$ would lead us immediately to Corollary 3 below.

Corollary 3. *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$B_{n+1} \leqq B_k \quad (k \geqq n+1). \tag{3.3}$$

Also let the function $f_{n+1}(z)$ *be defined by*

$$f_{n+1}(z) = z - \frac{e^{in\theta}}{(n+1)B_{n+1}} z^{n+1} \quad (f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)).$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^\lambda f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^\lambda f_{n+1}(z)\right|^\mu d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0).$$

The hypothesis (3.3) in Corollary 3 is weaker than the corresponding hypothesis in an earlier result of Sekine *et al.* [6, p.953, Theorem 6].

Next, for $p = 1$ and $q = 0$, Theorem 4 reduces to an integral means inequality of Sekine and Owa [5. Theorem 7] which, for $h = n$, yields another result of Sekine *et al.* [6, p.953, Theorem 7] under weaker hypothesis as mentioned above.

Finally, by setting $p = q = 1$ in Theorem 4, we obtain a slightly improved version of another integral means inequalities of Sekine and Owa [5, Theorem 8] with respect to the parameter λ (see also Sekine *et al.* [6, p.955, Theorem 8] for the case when $h = n$, just as remarked above).

References

[1] S.K. Chatterjea, On starlike functions, *J. Pure Math.* **1** (1981), 23-26.

[2] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* (2) **23** (1925), 481-519.

[3] S. Owa, On the distortion theorems. I, *Kyungpook Math. J.* **18** (1978), 53-59.

[4] T. Sekine and S. Owa, Coefficient inequalities for certain univalent functions, *Math. Inequal. Appl.* **2** (1999), 535-544.

[5] T. Sekine and S. Owa, On integral means inequalities for generalized subclasses of analytic functions, Proceedings of the Third ISAAC Congress, Berlin, August 2001.

[6] T. Sekine, K. Tsurumi, and H.M. Srivastava, Integral means for generalized subclasses of analytic functions, *Sci. Math. Japon.*, **54** (2001), 489-501.

[7] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**(1975), 109-116.

[8] H.M. Srivastava and S. Owa(Editors), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press(Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.

[9] H.M. Srivastava, S. Owa, and S.K. Chatterjea, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ. Padova* **75**(1987), 115-124.

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